Function: Conceptual Understanding and Connections with Patterns

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The proposed creation of a network domain model is implicitly encouraged in the Junior Certificate Mathematics syllabus which states that “where appropriate, connections should be made within and across the strands and with other areas of learning” and that “in each strand, and at each syllabus level, emphasis should be placed on making connections between the strands…” (Department of Education and Skills, 2016, pp. 10, 11).

It is accepted by mathematics educators that skills and procedural facility as well as conceptual understanding are essential in the teaching and learning of mathematics. However, there are arguments about which should be taught first: skills or concepts? Is the understanding of concepts dependent on the development of skills or is skill development predicated on conceptual understanding? (Dubinsky & Wilson, 2013). It would seem then that the building of a domain model requires careful consideration of how concepts and skills could and should be sequenced. Little-Johnson and Siegler (2001) accept that the developmental precedence of one type of knowledge over another has been hotly debated in the field of mathematics education. The results of their experimental study proposed an iterative model whereby “…conceptual and procedural knowledge develop iteratively, with increases in one type of knowledge leading to increases in the other type of knowledge, which trigger new increases in the first…” (Little-Johnson & Siegler, 2001, p. 346).

New mathematics syllabi for Junior and Leaving Certificate Mathematics were developed in Ireland between 2008 and 2015. This initiative, known as 'Project Maths', represented the first ‘root and branch’ revision of the post-primary mathematics curriculum since the implementation of the ‘New’ Mathematics curriculum between 1964 and 1973. (Shiel & Kelleher, 2017). The Project Maths Development Team was set up under this initiative. Part of its remit was to provide training for teachers in the newly recommended pedagogies for teaching Project Maths. During this training, many teachers requested resources to help them teach algebra and functions. Arising from these requests, the Maths Development Team (the word 'Project' has been dropped) designed and wrote three interconnected resource books: *Algebra Through the Lens of Functions Part 1 and 2* and *Student Workbook: Working on growing visual patterns: Linear Patterns Pack*. (Maths Development Team, 2016a, 2016b, 2016c). These resource books make connections between Strands 2, 4 and 5 of the Junior Certificate Mathematics syllabus.

As well as there being an a priori case to be made for connecting patterns (in Strand 4) and functions (in Strand 5), these connections are highlighted in the syllabus document which states that Strand 5 “... seeks to make explicit the connections and relationships already encountered in strand 3 and strand 4.” (Department of Education and Skills, 2016, p. 30)

Dubinsky and Wilson (2013) describe the "function concept" as "one of the most important topics in high school mathematics.” (p. 84). They suggest that while tests are effective in measuring skills and procedural knowledge, conceptual understanding is much harder to evaluate. Since a domain model is a hierarchical sequence of concepts, it seems reasonable to suggest that the sequencing of 'sub-concepts’ of the “function concept”, as well as the connected concepts of sequences in Strand 4 (Algebra), is important. In the United States, algebra is regarded as a very important ‘subject’ in its own right. There are a number of references in the literature to algebra serving as a "gatekeeper" subject (Kamii, 1990; Moses, Kamii, Swap & Howard, 1990). The word "gatekeeper" is used, as algebra is required for access to preparatory mathematics for many colleges in the United States.

Piaget et al (1968/1977) carried out one of the earliest investigations of understanding the concept of function. He applied his theory of reflective abstractions to linear functions, proportions and relations. Several authors subsequently published a variety of alternative theoretical perspectives including Vinner and Hershkowitz (1980), Sierpinska (1992), Bakar and Tall (2001), and Biehler (2005). Piaget ruminated on the origin of functions and postulated that there are two types/sources of functions: physical functions (causal) and logico-mathematical functions (operatory).

At its core, a "...function essentially expresses a dependence…" (Piaget et al, 1968/1977, p. 167). From a physical perspective, a function can be drawn from the observation of facts and the causal link immediately understood (e.g. the lengthening of a spring as a function of weight). Physical data are drawn from *objects*. From an operational perspective, numerical differences can be expressed as functions (e.g. having two piles of tokens, pile A and pile B, and adding two tokens to pile B for every token added to pile A). Piaget calls the links between A and B logico-mathematical links, which he says are the result of *actions* by a subject. The relationship between A and B only exists as a result of comparison by the subject. There is no physical or causal link.

Piaget sought to understand what characteristics are common to the multiple sources from which functions derive. He notes that space is a unifying concept whereby we find a mixture of both the physical and operatory. "Physical space is occupied by bodies and is also constructed operatorily by the subject." (Piaget et al, 1968/1977, p. 169). "Psychologically, the common source of operations and of causality is constituted by the actions of the self whose dynamic aspects enable the subject to experience, through simple abstractions, the first links to become causal, while the structures of their coordination give way to reflective abstractions thanks to which operations are constructed." (Piaget et al, 1968/1977, p. 170).

Dubinsky and Wilson (2013) have divided the literature on student conceptual difficulties in understanding the concept of function as follows:

(a) Knowing what is, and what is not, a function.
(b) Understanding the one-to-one property.
(c) Vertical line test.
(d) Representations of functions.
(e) Functional notation.
(f) APOS stages. Note: APOS theory was introduced in Cottrill et al. (1996) and was strongly influenced by the work of Piaget et al. (1968/1977).
(g) Composition of functions.

References

Bakar, M., & Tall, D. (1991). Students' mental prototypes for functions and graphs. In *Proceedings of the conference of the international study group for mathematics education (PME) (15th, Assisi, Italy, June 29–July 4), Vol. 1* (pp. 104–111).

Biehler, R. (2005). Reconstruction of meaning as a didactical task: The concept of function as an example. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 61–82). New York, NY: Springer.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior, 15*(2), 167–192.

Department of Education and Science. (2016). *Junior Certificate Mathematics Syllabus*. Retrieved from http://www.curriculumonline.ie/getmedia/4f6cba68-ac41-485c-85a0-32ae6c3559a7/JCSEC18_Maths_Examination-in-2016.pdf.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*(1), 83-101. https://doi.org/10.1016/j.jmathb.2012.12.001

Kamii, M. (1990). Opening the algebra gate: Removing obstacles to success in college mathematics courses. *The Journal of Negro Education, 59*(3), 392–406.

Maths Development Team. (2016a). Algebra Through the Lens of Functions Part 1. Dublin: Drumcondra Centre. Retrieved from http://www.projectmaths.ie/documents/pdf/AlgebraThroughTheLensOfFunctions.pdf

Maths Development Team. (2016b). Algebra Through the Lens of Functions Part 2. Dublin: Drumcondra Centre. Retrieved from http://www.projectmaths.ie/wp-content/uploads/2016/04/AlgebraThroughTheLensOfFunctionsPart2.pdf

Maths Development Team. (2016c). Student Workbook: Working on growing visual patterns: Linear Patterns Pack. Dublin: Drumcondra Centre. Retrieved from http://www.projectmaths.ie/documents/PDF/LinearPatternsWorkbook.pdf

Moses, R., Kamii, M., Swap, S., & Howard, J. (1989). The Algebra Project: Organizing in the spirit of Ella. *Harvard Educational Review, 59*(4), 423–443.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346-362. https://doi.org/10.1037/0022-0663.93.2.346

Shiel, G., & Kelleher, C. (2017). An Evaluation of the Impact of Project Maths on the Performance of Students in Junior Cycle Mathematics. Dublin: Educational Research Centre. Retrieved from http://www.erc.ie/wp-content/uploads/2017/11/PM_EvaluationStrand1.pdf

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel, & E. Dubinsky (Eds.), *The concept of functions: Aspects of epistemology and pedagogy* (pp. 25–28). United States: The Mathematical Association of America.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the fourth international conference for the psychology of mathematics education* (pp. 177–184). Berkeley: University of California, Lawrence Hall of Science.